ASYMPTOTIC SOLUTION OF A LINEAR NONHOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATION WITH A TRANSITION POINT AND ITS APPLICATION TO THE COMPUTATIONS OF TOROIDAL SHELLS AND PROPELLER BLADES

(ASIMPTOTICHESKOE RESHENIE LINEINOGO NEODNORODNOGO DIFFERENTSIAL'NOGO URAVNENIIA VTOROGO PORIADKA S PEREKHODNOI TOCHKOI I EGO PRILOZHENIIA K RASCHETAM TOROOBRAZNYKH OBOLOCHEK I LOPASTEI)

PMM Vol.23, No.6, 1959, pp. 1083-1094

S.A. TUMARKIN
(Moscow)

(Received 21 April 1959)

Differential equations of the type (1.1) containing a small parameter have found applications in many different problems.

The small parameter in the coefficient of the highest derivative corresponds to the presence of a boundary layer or some boundary effect. In such cases the approximate methods usually employed are inapplicable for practical purposes (i.e. such methods as the use of power or trigonometric series, interpolations and others). In the application of the asymptotic method, the presence of the small parameter does not lead to difficulties; on the contrary, it brings about simplifications. The process of numerical integration of a differential equation can be eliminated, because the general solution can be expressed in terms of known tabulated functions— exponential, Bessel functions and others.

In the use of the asymptotic solution one meets difficulties if the coefficient $q(x)$ in (1.1) has a simple zero within or on the boundary of the interval under consideration. (This is the case, for example, in the analysis of torus-like shells containing sections where the normal to the meridian is parallel to the axis of rotation, and also in the analysis of pipes with curvilinear axes. An example where the coefficient becomes zero on the boundary of the interval is presented by the case of study of curved shells, when the coefficient $q(x)$ vanishes due to the fact that in the edge section of the shell the centrifugal force vanishes. The same situation occurs in the study of rods).

In these cases, the ordinary asymptotic solutions which can be expressed by means of $e^{\pm x}$ are inapplicable. For the solutions of the homogeneous equation one obtains asymptotic series which contain Airy functions in place of $e^{\pm x}$ [1–8].

In the present article there are considered asymptotic solutions of non-homogeneous equations and some of their applications [8]. We note that some results along this line have been published in the works [9,10, 11].

The method presented makes it possible to obtain solutions in cases where $q(x)$ has a zero of arbitrary order.*

In Section 1 the equation (1.1) is reduced to a standard form; in Section 2 there is derived an asymptotic series for the solution of the homogeneous equation for the case when the small parameter has complex values; in Section 3 there are given contour integrals, power and asymptotic series for special functions entering solutions; in Section 4 there is considered the case of real values of the small parameter (this is a singular case); in Section 5, a second form is given for the asymptotic solution of the non-homogeneous equation; in Sections 6 and 7 there are given examples of the application of the method of computations of torus-like shells and rods and plates; in Section 8 there are given tables and graphs of functions involved in the solution.

1. Let us consider the differential equation

$$\varepsilon \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \varepsilon r(x)]\, y = f(x) \tag{1.1}$$

where ϵ is a small parameter, real or complex; the independent variable x is real; the coefficients in the equation are real functions possessing the necessary number of derivatives; the unknown function y and the right-hand part f may be complex-valued. In regard to the coefficients p and q it is assumed that $p(x) \neq 0$, and $q(x)$ becomes zero at one point only of the given interval (a, b). We may assume that this zero occurs at $x = 0$, where $q(x) = xq_1(x)$ and $q_1(x) \neq 0$.

---

* After the author had worked out the method presented here, he discovered the works of Reissner and Clark [9,10] treating the non-homogeneous equation

$$z'' - i\mu z \sin \xi = i\mu k \cos \xi$$

and related equations. A comparison shows that the method of Reissner and Clark is different from ours and that their solution is less complete and precise than ours. The precision of their solution is, in particular, lower than the precision of the equations of the theory of shells (see below, Sections 2 and 6).

Without restricting the generality, one may assume that $p(x)$ and $q_1(x)$ are positive.

The equation (1.1) can be reduced to the form [1,2, 4– 7]

$$\varepsilon \frac{d^2\eta}{du^2} + u\eta = g \tag{1.2}$$

that is, to the form of the well known equation

$$\frac{d^2\eta}{dt^2} + t\eta = 0 \tag{1.3}$$

with a right-hand side. For this purpose we replace the independent variable x in (1.1) by u, where u is the power series [5] in the parameter

$$u = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \ldots \tag{1.4}$$

while the function y is replaced by η by means of the substitution

$$y = w\eta, \qquad w = \frac{1}{\sqrt{pu'}} \tag{1.5}$$

where w is selected in such a way as to eliminate $d\eta/du$.

The primes indicate differentiation with respect to x. The coefficients of the series (1.4) are determined by quadratures from recurrence differential equations. The expressions for the first two coefficients are

$$u_0 = \pm\left(\frac{3}{2}\int_0^x \sqrt{\frac{|q|}{p}}\, dx\right)^{2/3} \tag{1.6}$$

where the sign of u_0 is chosen the same as that of x, and

$$u_1 = \frac{1}{2u_0^{1/2}}\int_0 \frac{\sigma}{u^{1/2}}\, dx \qquad \left(\sigma = v\,[rv + (pv')'],\ v = \frac{1}{\sqrt{pu_0'}}\right) \tag{1.7}$$

The right-hand side of equation (1.2) is equal to

$$g = \frac{f}{pwu'^2} = \frac{f}{\sqrt{pu'^3}} = \frac{fw}{u'} \tag{1.8}$$

where the function g can be represented in the following form because of (1.4):

$$g = g_0(x) + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \ldots \tag{1.9}$$

We note the useful equations:

$$pu_0u_0'^2 = q, \qquad pqv^4 = u_0, \qquad pv^2u_0' = 1 \tag{1.10}$$

By taking one, two, three or more terms of the series (1.4), we find that the solutions of equation (1.2) yield asymptotic approximations of higher and higher order for the equation (1.1) [5].

Let us introduce in place of x the complex independent variable

$$t = \rho u \qquad \left(\rho^3 = \frac{1}{\varepsilon}\right) \tag{1.11}$$

Equation (1.2) will then take the form

$$\frac{d^2\eta}{dt^2} + t\eta = \rho g \tag{1.12}$$

For small values of ϵ, the quantity g on the right-hand side of (1.12) will be a slowly changing function of t, because it changes as a function of $1/\rho$, as can be seen from (1.9) and (1.11).

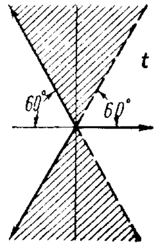

Fig. 1.

The solutions of the homogeneous equation (1.3) are Airy functions. Tables exist for these functions [2,3, 12–14].

2. We shall find an asymptotic series for a particular solution of the equation (1.12) with a slowly changing right-hand side along the straight line $t = \rho u$ of the complex plane t. These straight lines pass through the origin $t = 0$ and through the point $t = \rho$. We shall call them ρ-lines.

For practical applications it is necessary to obtain an asymptotic series for that unique solution, which, outside the neighborhood $x = 0$, is near the solution of the degenerate equation obtained by setting $\epsilon = 0$. The advantage of such a solution arises in connection with the boundary effect, because it makes it possible to satisfy the individual boundary conditions without the introduction of small differences.

The solutions of equation (1.12) or (1.2), by the method of the variation of the arbitrary constants, can be represented in the form

$$\eta = \frac{\rho^2}{W}\left[h_2(\rho u)\int_\alpha^u h_1(\rho\xi)\, g(\xi)\, d\xi - h_1(\rho u)\int_\beta^u h_2(\rho\xi)\, g(\xi)\, d\xi\right] \tag{2.1}$$

where $h_1(t)$, $h_2(t)$ constitute a fundamental system of solutions of (1.3). For the Wronskian we have the result

$$W = h_1\frac{dh_2}{dt} - h_2\frac{dh_1}{dt} = \text{const} \tag{2.2}$$

From the asymptotic expressions of Airy functions it follows that along any ρ-line, except for three lines containing the three singular rays (Fig. 1)

$$\arg t = 0, \qquad \tfrac{2}{3}\pi, \qquad \tfrac{4}{3}\pi \tag{2.3}$$

there exist solutions h_1 and h_2 that increase in absolute value in opposite directions according to a law given approximately by

$$|t|^{-1/4}\exp(\pm c|t|^{3/2})$$

where c is a constant depending on $\arg t$.

Along the singular rays every solution tends to zero. The ρ-lines contain singular rays if and only if the parameter $\epsilon = \rho^{-3}$ is real. Let us first consider the case of a complex ϵ. In Section 4 we shall return to the case of a real ϵ.

We denote by $h_1(t)$ that solution which increases as t changes from $t = 0$ to $t = \rho$. In the integrals (2.1) $h_1(\rho\xi)$ and $h_2(\rho\xi)$ are rapidly changing functions because ρ is a large parameter.

Taking for $g(\xi)$ Taylor's formula (or series)

$$g(\xi) = \sum \frac{(\xi-u)^n}{n!} g^{(n)}(u) \tag{2.4}$$

we obtain the final expression of the particular solution of the equation (1.2) in the form of a power series in $1/\rho = \epsilon^{1/3}$:

$$H = \rho e_0(\rho u)\, g(u) + e_1(\rho u)\, g'(u) + \frac{1}{\rho} e_2(\rho u)\, g''(u) + \ldots =$$

$$= \rho \sum_{n=0,1,2,\ldots} \frac{1}{\rho^n} e_n(\rho u)\, g^{(n)}(u) \tag{2.5}$$

where

$$e_n(t) = \frac{1}{n!\,W}\left[h_2(t)\int_{-\infty\rho}^{t} h_1(\tau)(\tau-t)^n\,d\tau + h_1(t)\int_{t}^{\infty\rho} h_2(\tau)(\tau-t)^n\,d\tau\right]$$

$$(n = 0, 1, 2, \ldots) \tag{2.6}$$

In particular, the first asymptotic approximation is equal to

$$H = \rho e_0(\rho u)\, g(u) \tag{2.7}$$

where

$$e_0(t) = \frac{1}{W}\left[h_2(t)\int_{-\infty\rho}^{t} h_1(\tau)\,d\tau + h_1(t)\int_{t}^{\infty\rho} h_2(\tau)\,d\tau\right] \tag{2.8}$$

The general solution will be

$$\eta = H + c_1 h_1 + c_2 h_2 \tag{2.9}$$

The functions $e_n(t)$ are completely determined on the ρ-lines and are the same for all equations (1.1); these functions are bounded on a ρ-line, since it follows from the asymptotic expressions of Airy functions h_1 and

h_2 that $e_0(t)$ $1/t$, e_1 $2/t^3$, ... as $t \to \infty$. The limits $-\infty, \rho$ and ∞, ρ in the integrals (2.6) are chosen for the purpose of obtaining bounded $e_n(t)$.

Since $y = w\eta$ by (1.5), by (2.9) we have obtained the general solution of the differential equation (1.1) in terms of Airy functions $h_1(t)$, $h_2(t)$ and the functions $e_n(t)$.

Tables and graphs of the functions $e_n(t)$ are given in Section 8.

Remarks 1. From equation (1.1) it can be seen that for small ϵ's the particular solutions will satisfy the formula y f/q, which fails to be valid only in the neighborhood of $x = 0$. The same result is obtained from formula (2.7) if one takes into consideration the asymptotic formula $e_0(t)$ t^{-1}, given in Section 3, and the formulas (1.5), (1.8) and (1.10); but the formula (2.7) does not cease to be valid in the neighborhood of $x = 0$.

2. When the parameter is absent from equation (1.1), i.e. when $\epsilon = 1$, the series (2.5), with $\rho = 1$, is of special interest because it represents the solution of the equation $\eta'' + t\eta = g(t)$.

3. The series (2.5) makes it possible to estimate the errors of the solution obtained for shells by Clark and Reissner [9,10]. The solution of Clark and Reissner differs from the first term of the series (2.5), but at the point $x = 0$ they are the same. Hence, in the solution of Clark and Reissner there is neglected a quantity of order $\epsilon^{1/3}$. This error is greater, for example, than the errors made in the standard theory of shells in which ϵ is proportional to the shell's thickness h, and where one usually takes into consideration not only $h^{1/3}$ but also $h^{1/2}$.

4. An estimate of the remainder term of the series (2.5) can be obtained either by means of (2.1) and (2.4), or directly from (2.5) making use of the boundedness of the functions $e_n(t)$. On the straight line $t = \rho u$ we have $|e_n(t)| < A$, where A is independent of n and t. From the expressions of $e_n(t)$ in terms of contour integrals (Section 3) it follows that on the straight line mentioned above, the values of $|e_n(t)|_{\max}$ tend to zero (as $n \to \infty$) more rapidly than the terms of any geometric progression, for the ratio of $|e_{n+1}(t)|_{\max}$ to $|e_e(t)|_{\max}$ decreases as $n^{-1/3}$.

The same holds for the derivatives $e_n'(t)$ and $e_n''(t)$. The series (2.5), therefore, converges for a large class of functions $g(u)$ and is a solution, because it formally satisfies the differential equation, as will be seen in Section 3, where there is also obtained an estimate of the remainder term of the series. If, for example, $|g^{(n)}(u)| < B^n$, where $B =$ const, then, dropping all terms from ρ^{-m} on, and taking the indicated coarse approximation $|e_n(t)| < A$, we obtain, for $|\rho| \geqslant 2B$, the upper bound of the remainder of the series

$$R_m = 2AB^{m+1}\rho^{-m}$$

This shows that the convergent series (2.5) also converges asymptotically.

3. 1. The functions $e_1(t)$, $e_2(t)$, ... can be expressed in terms $e_0(t)$. From (2.6) one can obtain, without difficulty,

$$(n+3)\,e_{n+3}(t) + te_{n+2}(t) + e_n(t) = 0 \qquad (n = 0, 1, 2, \ldots) \tag{3.1}$$

This yields the equations

$$\begin{aligned} e_1(t) &= 1 - te_0(t) \\ e_2(t) &= -\frac{1}{2}te_1(t) = \frac{1}{2!}[t^2e_0(t) - t] \\ e_3(t) &= \frac{1}{3!}[-(t^3+2)\,e_0(t) + t^2] \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{3.2}$$

2. If in (1.12) and (2.5) we select $g = 1, t, t^2, \ldots$, we see that $e(t)$, $e_1(t)$, ... satisfy the differential equations

$$\begin{aligned} &e_0'' + te_0 = 1 \\ &e_1'' + te_1 = -2e_0' \\ &e_2'' + te_2 = -(2e_1' + e_0) \\ &\ldots\ldots\ldots\ldots\ldots \\ &e_n'' + te_n = -(2e_{n-1}' + e_{n-2}) \quad (n = 2, 3, 4, \ldots) \end{aligned} \tag{3.3}$$

If one substitutes the series (2.5) into equation (1.2) one finds that the coefficients of all powers of $1/\rho$ vanish because of equations (3.3).

These equations are direct consequences of (2.6). Conversely, the differential equations (3.3) can serve for the determination of the functions $e_n(t)$; the functions $e_n(t)$ defined by the formulas (2.6) are the unique solutions of equations (3.3) which are bounded on the entire line $t = \rho u$, for all other solutions are obtained by the addition of $c_1h_1 + c_2h_2$.

3. Let us use the notation $L(\eta) = \eta''(t) + t\eta$. The functions $e_0(t)$, 1, t, t^2, t^3, ... are particular solutions of the equations

$$L(\eta) = 1,\ \ t,\ \ t^2,\ \ t^3 + 2\cdot 1,\ \ t^4 + 3\cdot 2t, \ldots \tag{3.4}$$

respectively.

In Section 5, the equations (3.4) will be used for the purpose of obtaining the asymptotic solution in a form different from that of the series (2.5).

4. Solving equations (3.3) by the method of Laplace, one can express the functions $e_n(t)$ as contour integrals.*

For the time being, while we are considering complex values of the parameter ϵ, let us select that value of $\rho = \epsilon^{-1/3}$ for which ρ, and hence also the line $t = \rho u$, lie within the double sector (Fig. 1)

$$\frac{1}{3}\pi < |\arg t| < \frac{2}{3}\pi \tag{3.5}$$

The solution of the first equation in (3.3) can be obtained in the form

$$e_0(t) = \int_0^\infty \exp\left(-tx - \frac{1}{3}x^3\right)dx \tag{3.6}$$

This integral coincides with (2.8) in the double sector (3.5), for this is the only solution which is bounded on the entire line $t = \rho u$.

The boundedness of the integral follows from the asymptotic series (3.11), which was obtained by the method of inversion, and which is valid in the sector $-2\pi/3 < \arg t < 2\pi/3$. The boundaries of this sector contain singular rays and correspond to the real values of the parameter ϵ which will be considered in the next section. Let us note that on the ray $t > 0$ the integral is finite and asymptotically equal to $1/t$, while on the ray $t \to -\infty$, the integral increases to infinity.

Analogously, we obtain

$$e_1(t) = \int_0^\infty x^2 \exp\left(-tx - \frac{1}{3}x^3\right)dx \qquad \text{etc.} \tag{3.7}$$

From these expressions it can be seen that

$$e_n(\bar{t}) = \overline{e_n(t)} \tag{3.8}$$

It is, therefore, sufficient to have tables for $e_n(t)$ in the sector (3.5) which lies in the upper half-plane.

---

* The function $e_0(t)$ is related to the generalized Airy integrals $\mathrm{Ei}_3(a)$ and $\mathrm{Si}_3(a)$ and to Lommel's functions [15]. An analogous function on the imaginary axis is used in the said article by Clark and Reissner [9]. The functions $\mathrm{Gi}(x)$ and $\mathrm{Hi}(x)$ which coincide with $e_0(t)$ on the real axis where used in problems other than those dealing with asymptotic solutions in the works [16–21].

From $e_0'' + te_0 = 1$ one finds that $e_0(t)$ is an infinite series with an infinite radius of convergence:

$$e_0(t) = e_0(0)\left(1 - \frac{1}{3!}t^3 + \frac{4}{6!}t^6 - \frac{4\cdot 7}{9!}t^9 + \frac{4\cdot 7\cdot 10}{12!}t^{12} - \cdots\right) + \\ + e_0'(0)\,t\left(1 - \frac{2}{4!}t^3 + \frac{2\cdot 5}{7!}t^6 - \frac{2\cdot 5\cdot 8}{10!}t^9 + \cdots\right) + \\ + t^2\left(\frac{1}{2!} - \frac{3}{5!}t^3 + \frac{3\cdot 6}{8!}t^6 - \frac{3\cdot 6\cdot 9}{11!}t^9 + \cdots\right) \tag{3.9}$$

In view of (3.6), the initial values are

$$e_0(0) = 3^{-2/3}\Gamma\left(\frac{1}{3}\right) = 1.287899, \quad e_0'(0) = -3^{-1/3}\Gamma\left(\frac{2}{3}\right) = -0.938893 \tag{3.10}$$

From (3.6) and (3.2) one obtains asymptotic series for large value of t on the lines $t = \rho u$ passing through the double sector (3.5):

$$e_0(t) \sim \frac{1}{t} + \sum_{k=1}^{\infty} \frac{(-1)^k (3k-1)!}{(k-1)!\,3^{k-1} t^{3k+1}} = \frac{1}{t} - \frac{2}{t^4} + \frac{40}{t^7} - \cdots$$

$$e_1(t) \sim \frac{2}{t^3} - \frac{40}{t^6} + \cdots, \qquad e_2(t) \sim -\frac{1}{t^2} + \frac{20}{t^5} - \cdots$$

$$e_3(t) \sim -\frac{6}{t^4} + \frac{360}{t^7} - \cdots \tag{3.11}$$

The series show that the functions $e_n(t)$ tend to zero as t goes to infinity. The graphs of the functions $e_n(t)$ are given in Section 8 for real and pure imaginary values of t.

4. If ϵ is real, one may select real values of $\rho = \epsilon^{-1/3}$. The values of $t = \rho u$ then lie on the real axis (Fig. 1) of the t-plane containing the singular ray $\arg t = 0$.

One can preserve the form of the solution (2.5) and the relations (3.2) and (3.3) for the functions $e_n(t)$. It is, however, not possible to preserve the asymptotic property $e_0(t) \sim 1/t$ on the entire real axis; one can, however, preserve this and the remaining asymptotic properties (3.11) separately for $t > 0$ and for $t < 0$ by taking for $e_0(t)$ different solutions of $e_0'' + te_0 = 1$, when $t > 0$ and $t < 0$. In case $t > 0$, we take as before, the solution (3.6). This is the only solution which decreases as $1/t$ when $t \to \infty$. Hereby the power series (3.9) is preserved with the initial conditions (3.10).

When $t < 0$, for $e_0(t)$ we select the solution

$$E_0(t) = \int_0^{\infty} \sin\left(t\xi - \frac{1}{3}\xi^3\right) d\xi \tag{4.1}$$

This is the only solution which decreases on the ray $t < 0$ as $1/t$, but other solutions are obtained, by the addition of solutions of the homogeneous equation, which decrease as

$$(-t)^{-1/4}\exp\left[-\frac{2}{3}(-t)^{3/2}\right]$$

The initial values of the series (3.9) are obtained from (4.1) and they differ from the values (3.10) by the factor $-1/2$:

$$\begin{aligned} E_0(0) &= -\frac{1}{2}3^{-2/3}\,\Gamma\left(\frac{1}{3}\right) = -0.643950 \\ E_0'(0) &= \frac{1}{2}3^{-1/3}\,\Gamma\left(\frac{2}{3}\right) = 0.469446 \end{aligned} \tag{4.2}$$

For the solution (4.1), i.e. not for the singular ray, the expression (2.8) remains valid; the functions $h_1(t)$ and $h_2(t)$ can be selected to be the Airy integrals $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. We should note that

$$\int_{-\infty}^{0}\mathrm{Bi}(x)\,dx = 0$$

With regard to the solution (3.6) it can be pointed out that it is connected to (4.1) by the relationship $e_0(t) - E_0(t) = \pi\,\mathrm{Bi}(-t)$. The properties (3.2), (3.11) and others of $e_n(t)$ remain valid for $E_n(t)$. The graphs of these functions are given in Section 8.

5. There exists another form of the asymptotic solution of the non-homogeneous equation. We rewrite equation (1.2) in the form

$$L(\eta) \equiv \varepsilon\frac{d^2\eta}{du^2} + u\eta = g, \qquad g(u) = \sum_{\nu=0}^{m} a_\nu u^\nu \tag{5.1}$$

Analogously to (3.4), one easily obtains

$$\begin{aligned} 1 = L[\rho e_0(\rho u)], \qquad u = L(1), \qquad u^2 = L(u) \\ u^3 = L[u^2 - 1\cdot 2\varepsilon\rho e_0(\rho u)] \\ u^4 = L(u^3 - 2\cdot 3\varepsilon) \quad \text{и т. д.} \end{aligned} \tag{5.2}$$

A particular solution of (5.1) is

$$\begin{aligned} H = \rho e_0(\rho u)\,(a_0 - 1\cdot 2a_3\varepsilon + 1\cdot 2\cdot 4\cdot 5a_6\varepsilon^2 - \cdots) + \\ + (a_1 + a_2u + a_3u^2 + \cdots) - \varepsilon(2\cdot 3a_4 + 3\cdot 4a_5u + 4\cdot 5a_6u^2 + \cdots) + \\ + \varepsilon^2(2\cdot 3\cdot 5\cdot 6a_7 + 3\cdot 4\cdot 6\cdot 7a_8u + \cdots) + \cdots \end{aligned} \tag{5.3}$$

This solution can be written in the form

$$\begin{aligned} H = \rho e_0(\rho u)\,[g(0) - \varphi_0''(0)\varepsilon + \varphi_1''(0)\varepsilon^2 - \cdots] + \\ + [\varphi_0(u) - \varphi_1(u)\varepsilon + \varphi_2(u)\varepsilon^2 - \cdots] \end{aligned} \tag{5.4}$$

Here

$$\varphi_0(u) = \frac{g(u) - g(0)}{u}, \quad \varphi_{n+1}(u) = \frac{\varphi_n''(u) - \varphi_n''(0)}{u} \quad (n = 0, 1, 2, \ldots) \quad (5.5)$$

The expressions (5.3) and (5.4) are power series in the parameter ϵ. The series (5.3) was given in another form (without the parameter but in powers of u) in a work by Miller and Zaki [17], where this series was considered as the solution of the equation $y'' - xy = \Sigma\, a_\nu x^\nu$ without any connection with the asymptotic solution of equation (1.1). The right-hand side g of the differential equation enters into the series (5.4) in the form of more complicated expressions than in the series (2.5), but in contrast with the series (2.5), the series (5.4) contains only $e_0(t)$.

6. As an illustrative example let us consider the application of our theory to the analysis of toroidal shells.

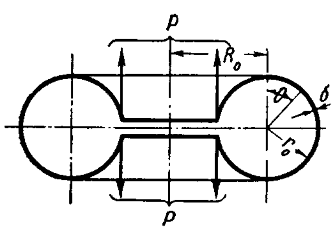

Fig. 2.

The solution of an asymptotic nature for complete toroidal shells was first given by Clark and Reissner [9, 10]. The combining into one solution of trigonometric series and Hankel's function [22, 23] is not advisable, because the accuracy of the solution in terms of Hankel's function is low for small values of the parameter ρ, while for large values and thin shells the evaluation of the trigonometric series is very cumbersome. This was pointed out in author's work [24], where it was also stated that Hankel's functions should be used for the asymptotic solution.

Let us now consider the toroidal shell (Fig. 2) cut along the parallel $\theta = -\pi/2$ and subjected to a tensile axial force P and to uniform, normal, internal pressure p_n [24]. The equations for the shell [22] can be written in the form

$$\varepsilon \frac{d}{d\theta}\left(p \frac{dV}{d\theta}\right) + qV = f \quad (6.1)$$

where V is the complex solution function,

$$p = \frac{1}{1 + \alpha \sin\theta}, \qquad q = \frac{\sin\theta}{(1 + \alpha \sin\theta)^2}, \qquad \alpha = \frac{r_0}{R_0}$$
$$\varepsilon = \frac{1}{\rho^3} = -\frac{i}{l^3}, \qquad \rho = -il, \qquad l^3 = \sqrt{12(1 - \mu^2)}\, \frac{r_0^2}{R_0 \delta} \quad (6.2)$$
$$f = \left[il^3\left(\frac{P}{2\pi r_0} + \frac{2 - \alpha}{2} R_0 p_n\right) - \frac{1}{2}\alpha r_0 p_n\right] \frac{\cos\theta}{(1 + \alpha \sin\theta)^2}$$

Here μ is Poisson's ratio, δ is the thickness of the shell, so that ρ^3 is a large parameter.

When $\theta = 0$, or $\theta = \pm \pi$, the function q has a simple zero, and one may apply the above derived asymptotic solution to equation (6.1).

The parameter $\rho = - il$ is pure imaginary. Therefore, one uses the graphs and tables of the functions $e_n(t)$ for pure imaginary t in the computations.

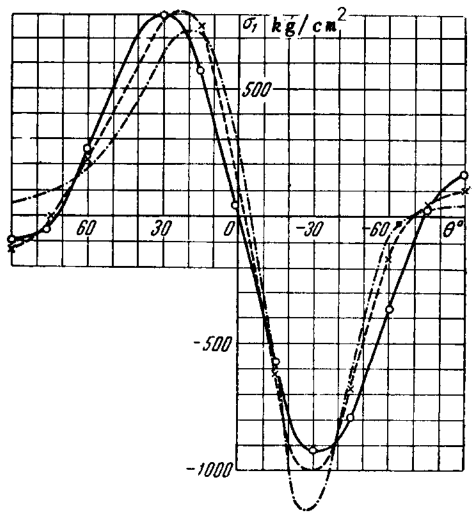

Fig. 3.

On the graph (Fig. 3) there are shown the results of the asymptotic computation of a steel shell with $R_0 = 16$ cm, $r_0 = 5$ cm, $\delta = 3$ mm, $P = 1000$ kg, $p_n = 0$. A comparison with other solutions is also shown; the solutions for which one or two terms of the series (2.5) were retained are shown by a dotted and by a solid curve respectively; the solution based on the Clark-Reissner [9] method is shown by a curve made of dots and of short line segments; the small circles along the solid curve show the solution obtained earlier [24] by means of trigonometric series.

The asymptotic solution (5.4), with the retention of the zeroth and first degree terms in ρ, coincides with the trigonometric series solution indicated by the small circles.

The investigation has yielded the following results:

1) The asymptotic analysis by the use of one or two terms of the series (2.5) or (5.4) gives quite good results, even for moderate values of the parameter ρ^3;

2) Considerably less accurate results are obtained by the method of Clark and Reissner (in which the relative error is of the order $1/\rho$, as was shown in Section 2).

In order to illustrate the suitability of the asymptotic method for obtaining of computational formulas, we introduce the formula for the deflection of a toroidal shell.

For a shell (Fig. 2) which is cut along the parallel $\theta = - \pi/2$, the separation of the cut edges, under the action of the axial force P and of the internal pressure p_n, is equal to

$$\Delta = -\frac{2R_0}{E\delta}\int_{-1/2\pi}^{1/2\pi}\frac{\cos\theta}{1+\alpha\sin\theta}\,\mathrm{Re}\,V d\theta$$

Considering the first term only in (2.5) and recalling that, for large ρ, $e_0(t)$ decreases rapidly in both directions from $\theta = 0$, we obtain the asymptotic expression for the deflection

$$\Delta \sim \frac{1}{E}\sqrt{12(1-\mu^2)}\,\frac{r_0}{\delta^2}\,[P + (2-\alpha)\,\pi r_0 R_0 p_n] \tag{6.3}$$

For given dimensions R_0 and r_0, and small enough shell thickness δ, this formula shows that the deflection is inversely proportional to the square of the shell's thickness.

If the pressure $p_n = 0$, then the deflection is proportional to the radius r_0 and is independent of R_0. Such results are of importance in the design of structures.

The asymptotic solutions (2.5) and (5.4) can also be applied to the investigation of pipes with curvilinear axis.

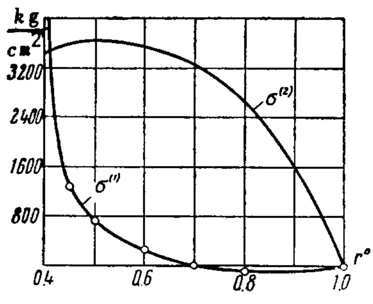

Fig. 4.

We call attention to the fact that in the derivation of the formula (6.3) it is necessary to compute (with the aid of (3.6)) the integral

$$\int_0^{\infty}\mathrm{Re}\,e_0(iy)\,dy = \int_0^{\infty}\left(\int_0^{\infty} e^{-\frac{1}{3}x^3}\cos xy\,dx\right)dy$$

On the basis of a property of the Fourier cosine transformation this integral is equal to $\pi/2$.

7. As a second example, we present the results of an investigation of a very thin curved steel blade rotating with great speed.

The differential equation of the curved blade is of the type (1.1); the role of the large parameter is played by the square of the angular velocity of the given blade [11,25].

For fast rotating blades the accuracy obtained from the first term of the series is sufficient, and the solution is given by simple formulas.

The curves in Fig. 4 show that for high angular velocities there arises a strong boundary effect. Near the clamped end of the blade there takes place a concentration of the bending stress $\sigma^{(1)}$; $\sigma^{(2)}$ is the tensile stress).

This indicates also that the curvature of the elastic curve is large near the clamped end. In the remaining part of the blade the elastic curve is near the special type of a cable-type, or chain-type curve, namely, it will approximate to the form of the elastic curve of a blade which does not resist bending relative to the axis of the minor rigidity of the cross-section.

In the work [26] it is stated that one can usually carry out the asymptotic analysis of blades by means of the function $e^{\pm x}$, avoiding the use of Airy functions. In the work [27] there is constructed such an asymptotic solution for propellers with variable pitch.

8. 1. In Fig. 5 there are given graphs of $e_n(t)$ and $e_n'(t)$ for pure imaginary values $t = iy$; the primes in the notations of the curves indicate differentiation with respect to y. The graphs serve for the asymptotic solution of equation (1.1) for pure imaginary values of the parameter ϵ (for example, for the computation of shells see Section 6). Re e_n and Im e_n denote the real and imaginary parts: $e_n = \mathrm{Re}\, e_n + i\mathrm{Im}\, e_n$. Furthermore, $de_n/dt = -\, i de_n/dy$. The graphs are given for positive y. For negative y, one can use the eveness of Re e_n and of d Im e_n/dy and the oddness of Im e_n and of d Re e_n/dy.

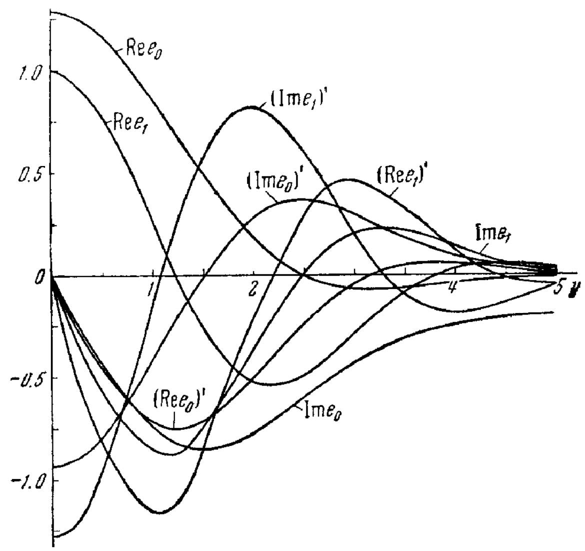

Fig. 5.

In the article [10] there is given a table of values which is equivalent to the $e_0(t)$ on the imaginary axis of t.

2. The graphs given in Fig. 6 represent $e_n(t)$ and $e_n'(t)$ for real

negative values of t. They are used, in particular, for the investigation and calculation of propeller blades (see Section 7).

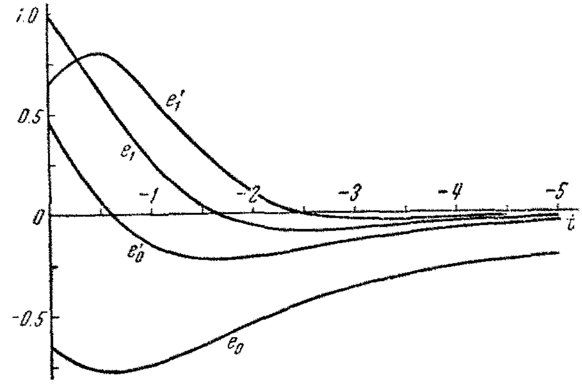

Fig. 6.

Tables which are equivalent to tables of e_0 on the real axis are given in the works [16÷21] where the following notations are used

$$e_0(t) = -\pi \mathrm{Gi}(-t)$$
$$E_0(t) = \pi \mathrm{Hi}(-t)$$

3. For the asymptotic solution of equation (1.1) one needs, in addition to the functions $e_n(t)$, in accordance with (2.9) also the Airy functions $h_1(t)$ and $h_2(t)$.

Their values are given in tables [14] for complex values of t. For real values of t one can use tables given in [2,3,12,13].

BIBLIOGRAPHY

1. Langer, R.E., On the asymptotic solutions of ordinary differential equations with an application to the Bessel functions of large order. *Trans. Amer. Math. Soc.*, Vol. 33, No. 1, 1931.

2. Fok, V.A., Difraktsiia radiovoln vokrug zemnoi poverkhnosti (Difraction of radio waves around the earth's surface). *Izd. Akad. Nauk SSSR*, Moscow-Leningrad, 1946.

3. Fok, V.A., Tablitsy funktsii Eiri (Tables of Airy Functions). *Inform. otd.* NII-108, Moscow, 1946.

4. Langer, R.E., The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point. *Trans. Amer. Math. Soc.*, Vol. 67, No. 2, 1949.

5. Cherry, T.M., Uniform asymptotic formulas for functions with transition points. *Trans. Amer. Math. Soc.*, Vol. 68, No. 2, 1950.

6. Dorodnitsyn, A.A., Asimptoticheskie zakony raspredeleniia sobstvennykh znachenii dlia nekotorykh osobykh vidov differentsial'nykh uravnenii vtorogo poriadka (Asymptotic laws of the distribution of the characteristic values for certain special types of differential equations of the second order). *Progr. math. Sci.*, Vol. 7, No. 6, Moscow, 1952.

7. Olver, F.W.J., The asymptotic solution of linear differential equations of the second order in domain containing one transition point. *Phil. Trans., Ser. A*, Vol. 249, No. 959, 1956.

8. Tumarkin, S.A., Asimptoticheskoe reshenie lineinykh neodnorodnykh differentsial'nykh uravnenii i ego prilozheniia k raschetu obolochek i lopastei (Asymptotic solution of linear non-homogeneous differential equations and its application to the investigation of shells and propeller blades). *Trudy Tret'ego vsesoiuznogo matem. s'ezda*, Moscow, June-July, 1956, IV, Akad. Nauk SSSR, Moscow, 1959.

9. Clark, R.A. and Reissner, E., Bending of curved tubes. *Advanc. Appl. Mech.* Vol. 2, 1951. There exists a Russian translation: *Problemy Mekhaniki*. Sb. st. IL (Collection of articles from Foreign literature), 1955.

10. Clark, R.A., On the theory of thin elastic toroidal shells. *J. Math. Phys.* Vol. 29, No. 3, 1950.

11. Tumarkin, S.A., Raschet lopastei vintov na izgib pri ves'ma bol'shikh skorostiakh vrashcheniia (asimptoticheskii metod) (Analysis of propeller blades with respect to bending under very high angular velocities (asymptotic method)). *MAP SSSR*, Trudy, No. 667, 1948.

12. Smirnov, A.D., Tablitsy funktsii Eiri i spetsial'nykh vyrozhdennykh gipergeometricheskikh funktsii dlia asimptoticheskikh reshenii differentsial'nykh uravnenii vtorogo poriadka (Tables of Airy functions and of special degenerate hypergeometric functions for asymptotic solutions of differential equations of the second order). *Akad. Nauk SSSR*, Moscow, 1955.

13. Miller, J.S.P., *The Airy Integral. Mathematical Tables*. Part-vol. B., Brit. Assoc., Cambridge, 1946.

14. Tables of modified Hankel functions of order one-third and their derivatives. *The Annals of the Computation Laboratory of Harvard University*, Vol. 2, 1945.

15. Watson, G.N., *Theory of Bessel Functions*, Pt. I. Russian translation IIL, Moscow, 1949.

16. Scorer, R.S., Numerical evaluation of integrals of the form.... *Quart. J. Mech.*, Vol. 3, Pt. 1, 1950.

17. Miller, J.S.P. and Zaki, Mursi, Notes on the solution of the equation $y'' - xy = f(x)$. *Quart. J. Mech.*, Vol. 3, Pt. 1, 1950.

18. Rothman, M., The problem of an infinite plate under an inclined loading, with tables of the integrals of Ai($\pm x$) and Bi($\pm x$). *Quart. J. Mech.*, Vol. 7, Pt. 1, 1954.

19. Rothman, M., Tables of the integrals and differential coefficients of Gi($+x$) and Hi($-x$). *Quart. J. Mech.*, Vol. 7, Pt. 3, 1954.

20. Tyler, S.M. and Christiano, J.G., An Airy integral analysis of beam columns with distributed axial loading having fixed line of action. *J. Appl. Mech.*, Vol. 19, No. 3, 1952.

21. Tyler, S.M. and Rouleau, W.T., An Airy integral analysis of beam columns with distributed axial loading that deflects with the column. *Proc. 2nd USA Congress of Appl. Mech.*, 1955.

22. Zenova, E.F. and Novozhilov, V.V., Simmetrichnaia deformatsiia toroobraznykh obolochek (Symmetric deformation of toroidal shells). *PMM* Vol. 15, No. 5, 1951.

23. Novozhilov, V.V., *Teoriia tonkikh obolochek (Theory of Thin Shells)*. Sudpromgiz, 1951.

24. Tumarkin, S.A., Raschet simmetrichno-nagruzhennykh toroobraznykh obolochek pri pomoshchi trigonometricheskikh riadov (Analysis of symmetrically loaded toroidal shells with the aid of trigonometric series). *PMM* Vol. 16, No. 5, 1952.

25. Birger, I.A., *Nekotorye matematicheskie metody resheniia inzhenernykh zadach (Some Mathematical Methods of Solving Engineering Problems)*. Oborongiz, Moscow, 1956.

26. Tumarkin, S.A., Printsipy postroeniia asimptoticheskogo metoda reascheta lopastei na izgib (Principles of construction of the asymptotic method of analysis of propeller blades in bending). *Nauchn. trudy Mosk. gornogo in-ta*. Sb. No. 23, Moscow, 1958.

27. Torchinskaia, E.L., Raschet na izgib v pole tsentrobezhnykh sil vysokooborotnykh lopastei peremennogo vintovogo shaga (asimptoticheskii method) (Analysis, with respect to bending in the field of centrifugal forces, of high speed propeller blades with variable pitch (asymptotic method)). *Nauchn. trudy Mosk. gornogo in-ta*. Sb. No.29, 1959.

Translated by H.P.